

$N = 2$ supersymmetric spin foams in three dimensions

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We construct the spin foam model for $N = 2$ supergravity in three dimensions. Classically, it is a BF theory with gauge algebra $\mathfrak{osp}(2|2)$. This algebra has representations which are not completely reducible. This complicates the procedure when building a state sum. Fortunately, one can and should excise these representations. We show that the restricted subset of representations form a subcategory closed under tensor product. The resulting state-sum is once again a topological invariant. Furthermore, within this framework one can identify positively and negatively charged fermions propagating on the spin foam. These results on $\mathfrak{osp}(2|2)$ representations and intertwiners apply more generally to spin network states for $N = 2$ loop quantum supergravity (in $3 + 1$ dimensions) where it allows to define a notion of BPS states.

I. INTRODUCTION

Supergravity and supersymmetry have enjoyed widespread popularity and attracted intensive study in the general high energy physics community. They have, however, been relatively under-appreciated in the arena of non-perturbative quantum gravity, with the exception of [1, 2, 3] on canonical side and [4] for spin foams in 3 dimensions.

In [4], they provide a general framework for the quantization of supersymmetric theories, with emphasis placed on $N = 1$ super- BF theory with gauge algebra $\mathfrak{osp}(1|2)$. This arises as a particular example of a super Chern-Simons theory, proposed classically in [5], in the limit where the cosmological constant goes to zero. The quantization technique is based upon the $\mathfrak{su}(2)$ Ponzano-Regge model for quantum gravity in three dimensions. It is a spin foam model regularizing in a cut-off independent way the BF -path integral. Furthermore, representations of $\mathfrak{osp}_E(1|2)$ ¹ are formed from the direct sum of two irreducible $\mathfrak{su}(2)$ representations. Therefore, the resulting $\mathfrak{osp}_E(1|2)$ model contains the $\mathfrak{su}(2)$ Ponzano-Regge state sum nested within. The other configurations, referred to as its superpartners, can be identified with fermions propagating along the edges of the spin foam. A plausible argument was given to this interpretation of fermions propagating and interacting on a dynamical background, but it was not developed explicitly. Only an asymptotic analysis of the spin foam amplitudes, and a subsequent comparison with configurations of super-Regge calculus would provide the necessary reinforcement. We shall review this in Section II.

Our quest here is not to investigate issues pertaining to the semi-classical regime. We leave that for later work. But we want to extend the formalism to the $N = 2$ scenario. Extended supersymmetry is essential to the success of string theory, which investigates quantum gravity in the perturbative regime. A consistent spin foam theory with $N = 2$ supersymmetry allows one to examine the properties of BPS states in a non-perturbative setting, and it could provide a way to compare results on the black hole entropy got by both string theory and loop quantum gravity. The difficulty facing us when trying to quantize $\mathfrak{osp}(2|2)$ BF theory is that the representation theory of the algebra is highly non-trivial. The representations are not all distinguishable using the two Casimirs. Moreover, some of these representations are not even completely reducible. Propitiously, excising these representations (among others), we arrive at a subcategory which is closed under tensor product, and which is exactly the subset of representations upon which one can define a star product [6]. Then such a category of representations stable under tensor product allows to define a topological state sum model [7]. As we will see, pure $N = 2$ supergravity naturally incorporates two

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¹ We differentiate the cases of Riemannian and Lorentzian supergravity theories with the subscripts E and L , respectively.

charged fermions and a (topological) $u(1)$ gauge field. Thus, the spin foam model will contain configurations interpretable as positively and negatively charged fermions propagating along its edges. Similar matters have recently come under investigation [8, 9, 10, 11, 12, 13], and have received much attention in the case of spin foam quantum gravity.

II. REVIEW OF $N = 1$ SUPERSYMMETRIC SPIN FOAMS

For a thorough investigation of $N = 1$ supersymmetric spin foams we refer the reader to [4]. We start from a BF -type action for $3d$ supergravity with zero cosmological constant

$$\mathcal{S}[\mathcal{E}, \mathcal{A}] = \int_{\mathcal{M}} \text{Str}(\mathcal{E} \wedge \mathcal{F}[\mathcal{A}]) \quad (1)$$

where \mathcal{E} is the supertriad, \mathcal{A} is the superconnection, while $\mathcal{F}[\mathcal{A}] = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ is the supercurvature. Both \mathcal{E} and \mathcal{A} are 1-forms valued in the super lie algebra, and Str is its supertrace. In the case of Riemannian supergravity, this gauge algebra is $\mathfrak{osp}_E(1|2)$. Its bosonic subalgebra is $\mathfrak{su}(2)$. It is a minimal supersymmetric extension of $3d$ gravity.

As a brief aside, we may easily add a positive cosmological constant term to the action and in this context the action is equivalent to a super Chern-Simons theory devised by Achúcarro and Townsend [5]. The theory in question is Riemannian deSitter supergravity with gauge algebra $\mathfrak{osp}_E(1|2) \times \mathfrak{osp}_E(1|2)$.

Apart from its $\mathfrak{su}(2)$ subalgebra, $\mathfrak{osp}_E(1|2)$ has fermionic (anti-commuting) generators Q_{\pm} . Together with J_{\pm} , J_3 , they satisfy the algebra

$$\begin{aligned} [J_3, J_{\pm}] &= \pm J_{\pm}, & [J_+, J_-] &= 2J_3, \\ [J_3, Q_{\pm}] &= \pm \frac{1}{2}Q_{\pm}, & [J_{\pm}, Q_{\pm}] &= 0, & [J_{\pm}, Q_{\mp}] &= Q_{\pm}, \\ \{Q_{\pm}, Q_{\pm}\} &= \pm \frac{1}{2}J_{\pm}, & \{Q_{\pm}, Q_{\mp}\} &= -\frac{1}{2}J_3 \end{aligned} \quad (2)$$

The supergravity fields written in terms of generators of the algebra are

$$\mathcal{E} = E^i J_i + \phi^A Q_A, \quad \mathcal{A} = W^i J_i + \psi^A Q_A, \quad (3)$$

where E and W are the triad and connection, while ϕ and ψ represent the fermion field. $A \in \{\pm\}$ and $i \in \{1, 2, 3\}$. The action may be rewritten in terms of these variables as

$$\mathcal{S}_{N=1}[E, W, \phi, \psi] = \int_{\mathcal{M}} \left\{ \text{Str}(E \wedge (F[W] + \psi \wedge \psi)) + \phi \wedge D\psi \right\} \quad (4)$$

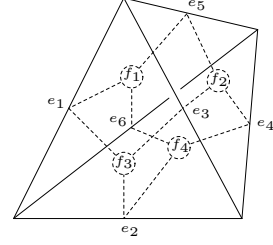
where $F(W) = dW + W \wedge W$ is the gravitational curvature, and we define the operator as $D = d + W \wedge$. This action describes a fermion field propagating on a manifold \mathcal{M} endowed with a dynamical geometry.²

The recipe for quantization bases itself on the Ponzano-Regge (PR) model for $3d$ Riemannian quantum gravity [14]. The PR model is a discrete state sum, derivable from the $\mathfrak{su}(2)$ - BF path integral. The method is to triangulate the manifold \mathcal{M} using a simplicial complex Δ . The gravitational information is encoded in the representations of $\mathfrak{su}(2)$ which label the edges e of Δ , denoted V^{j_e} . The amplitude assigned to the triangular faces f of each tetrahedron in Δ are invariant tensors. They are the $\mathfrak{su}(2)$ $3j$ -symbols intertwining the three edge representations of the triangle, denoted $i_f : V^{j_1} \otimes V^{j_2} \otimes V^{j_3} \rightarrow \mathbb{C}$ or $i_f : V^{j_1} \otimes V^{j_2} \otimes (V^{j_3})^* \rightarrow \mathbb{C}$ depending on the relative orientation of the edges.³ The partition function arises by applying this procedure to a closed manifold \mathcal{M} , and takes the form

² The spinor indices follow the north-west convention so that $\phi^A = \epsilon^{AB} \phi_B$ and $\phi_A = \phi^B \epsilon_{BA}$. The metric on the spinor space is the anti-symmetric tensor ϵ_{AB} with $\epsilon_{+-} = \epsilon^{+-} = 1$. This implies $\epsilon^{AB} \epsilon_{BC} = -\delta^A_B$. The quadratic Casimir of $\mathfrak{osp}_E(1|2)$ determines its supertrace. This takes the form $C_2 = J_i \eta^{ij} J_j + Q_A \epsilon^{AB} Q_B$, where $\eta_{ij} = \delta_{ij}$. Thus $\text{Str}(J_i J_j) = \eta_{ij}$, $\text{Str}(Q_A Q_B) = -\epsilon_{AB}$, and $\text{Str}(J_i Q_A) = 0$.

³ The state sum includes both orientations for each edge.

$$\mathcal{Z}_{BF}[\Delta] = \sum_{V^{je}} \prod_e \dim(V^{je}) \prod_t \mathcal{T}(V^{je}), \quad \text{where} \quad \mathcal{T}(V^{je}) =$$



where e and t are the edges and tetrahedra of Δ respectively. V^{je} is the irreducible representation of $\mathfrak{su}(2)$ labeled by $j_e \in \frac{1}{2}\mathbb{N}$. $\dim(V^{je}) = 2j_e + 1$ is the dimension of V^{je} . Finally, $\mathcal{T}(V^{je}) = \{6j\}_{\mathfrak{su}(2)}$ is the $\{6j\}$ -symbol for $\mathfrak{su}(2)$. It is the amplitude for the tetrahedron and it arises from a contraction of the intertwiners assigned to the four faces which bound the tetrahedron t . In (5), we also drew the contracted intertwiners as a trivalent graph on the boundary of the tetrahedron. This is known as the boundary spin network. Another important structure is the dual 2-skeleton of the triangulation denoted Δ_2^* . We can equivalently think of the representations as labeling the faces f^* of Δ_2^* and the intertwiners as labeling the edges e^* of Δ_2^* . This is the structure known as a spin foam.

The resulting amplitude is a topological invariant, and thus is independent of the particular triangulation one chooses initially for \mathcal{M} .⁴ This is as it should be, since $3d$ gravity lacks local degrees of freedom propagating in the bulk.

One hopes naïvely, that by swapping $\mathfrak{su}(2)$ for $\mathfrak{osp}_E(1|2)$ one can arrive at a topological state-sum for $N = 1$ quantum supergravity. The representations of $\mathfrak{osp}_E(1|2)$ can be decomposed over its bosonic subalgebra. Each representation R^j of $\mathfrak{osp}_E(1|2)$ comprises of the direct sum of two representations of $\mathfrak{su}(2)$.⁵ Thus the representations of $\mathfrak{osp}_E(1|2)$ are once again labeled by $j \in \frac{1}{2}\mathbb{N}$ with $j \geq \frac{1}{2}$. Furthermore, the tensor product of two representations of $\mathfrak{osp}_E(1|2)$ satisfy a rule analogous to that of $\mathfrak{su}(2)$ except that the sum over j goes in *half-integer* steps

$$R^j = V^j \oplus V^{j-\frac{1}{2}}, \quad R^{j_1} \otimes R^{j_2} = \bigoplus_{|j_1-j_2| \leq j \leq j_1+j_2} R^j. \quad (7)$$

In the case of $\mathfrak{osp}_E(1|2)$, we cannot simply define a star operator and talk about unitary representations. To construct unitary representations, we need a grade star operator. We obtain this by labeling each irreducible representation R^j with a parity $\lambda = 0, 1$, so that the representation is decomposed as $R^{j,\lambda} = V^{j,\lambda} \oplus V^{j-\frac{1}{2},\lambda+1}$. As it stands, we have two copies of each representation, one for each choice of parity, but if the subalgebra $\mathfrak{su}(2)$ is to play the physical role of rotations, the representations must obey the spin-statistics relation of quantum field theory. In other words, the V^j should be even or odd depending on whether j is an integer or not. The implications are

$$\begin{aligned} \text{For } j \in \mathbb{N}, \quad & Q_+^\dagger = -Q_-, \quad \text{and} \quad Q_-^\dagger = Q_+, \\ \text{for } j \in \mathbb{N} + \frac{1}{2}, \quad & Q_+^\dagger = Q_-, \quad \text{and} \quad Q_-^\dagger = -Q_+. \end{aligned}$$

⁴ Often we are interested in transitions amplitudes between two quantum states. For this we need a manifold with boundary \mathcal{S} . A quantum state is a spin network representing the gravitational information residing on the boundary \mathcal{S} . A spin network is a graph labeled with the boundary gravitational information, and is dual to a triangulation $\Delta_{\mathcal{S}}$ of \mathcal{S} . Then, Δ should coincide with $\Delta_{\mathcal{S}}$ on the boundary. The resulting amplitude is a topological invariant up to the boundary contributions.

⁵ The action of the operators on the representation R^j is

$$\begin{aligned} J_3|j, j, m\rangle &= m|j, j, m\rangle, & J_3|j, j - \frac{1}{2}, m\rangle &= m|j, j, m\rangle, \\ J_{\pm}|j, j, m\rangle &= \sqrt{(j \mp m)(j \pm m + 1)}|j, j, m \pm 1\rangle, & J_{\pm}|j, j - \frac{1}{2}, m\rangle &= \sqrt{(j - \frac{1}{2} \mp m)(j + \frac{1}{2} \pm m)}|j, j - \frac{1}{2}, m \pm 1\rangle, \\ Q_{\pm}|j, j, m\rangle &= \mp\sqrt{j \mp m}|j, j - \frac{1}{2}, m \pm \frac{1}{2}\rangle, & Q_{\pm}|j, j - \frac{1}{2}, m\rangle &= -\frac{1}{2}\sqrt{j + \frac{1}{2} \pm m}|j, j, m \pm \frac{1}{2}\rangle \end{aligned} \quad (6)$$

while the Casimir of the representation is $C^j = j(j + \frac{1}{2})$.

It is this parity issue that complicates the simple replacement of $\mathfrak{su}(2)$ by $\mathfrak{osp}_E(1|2)$. Unlike the $\mathfrak{su}(2)$ case, the $3j$ -symbols of $\mathfrak{osp}_E(1|2)$ no longer have a purely combinatorial definition, but also depend on their graphical representation. This dependence is familiar from quantum groups. The appropriate dependence can be incorporated into a graphical calculus known as circuit diagrams described in [4, 7].⁶ Taking into account this dependence, we arrive at the state sum model

$$\mathcal{Z}_{N=1}[\Delta] = \sum_{R^{je}} \prod_e \text{Sdim}(R^{je}) \prod_t \mathcal{T}(R^{je}), \quad (8)$$

where $\mathcal{T}(R^{je}) = \{6j\}_{\mathfrak{osp}_E(1|2)}$ is the $\{6j\}$ -symbol for $\mathfrak{osp}_E(1|2)$. The superdimension of the representation is given by the supertrace of the identity element in that representation

$$\text{Sdim}(R^j) = (-1)^{2j} \dim(V^j) - (-1)^{2j} \dim(V^{j-\frac{1}{2}}) = (-1)^{2j}. \quad (9)$$

A representation of $\mathfrak{osp}_E(1|2)$ labels each edge. The four triangles of each tetrahedron in Δ are each labeled by a trivalent intertwiner I_f of $\mathfrak{osp}_E(1|2)$. A contraction of these intertwiners gives rise to $\{6j\}_{\mathfrak{osp}_E(1|2)}$. This defines a topological state sum as proved in [4] following the circuit diagram techniques introduced in [7]. It is interpreted as providing the spinfoam quantization of $N = 1$ 3d supergravity.

We shall be more precise. To construct the amplitude, we first label each edge of Δ with an orientation. This allows us to distinguish a representation from its dual. (In a PR-like model, one sums over both orientations.) Then we consider a tetrahedron t in Δ . We assign an intertwiner to each of its faces: $I_f : R^{j_1} \otimes R^{j_2} \otimes R^{j_3} \rightarrow \mathbb{C}$ or $I_f : R^{j_1} \otimes R^{j_2} \otimes (R^{j_3})^* \rightarrow \mathbb{C}$ depending on the relative orientation of the three edges of that face. This intertwiner is unique up to normalisation and we may normalise it by imposing that the complete contraction of two intertwiners equals the identity. Contracting these four intertwiners produces the $\{6j\}_{\mathfrak{osp}_E(1|2)}$ amplitude for the tetrahedron.

Applying the isospin decomposition to the intertwiner of $\mathfrak{osp}_E(1|2)$ allows us to distinguish between two cases, as shown in Figure 1. Four of these intertwiners satisfy $j_1 + j_2 + j_3 \in \mathbb{N}$. These occur in the $\mathfrak{su}(2)$

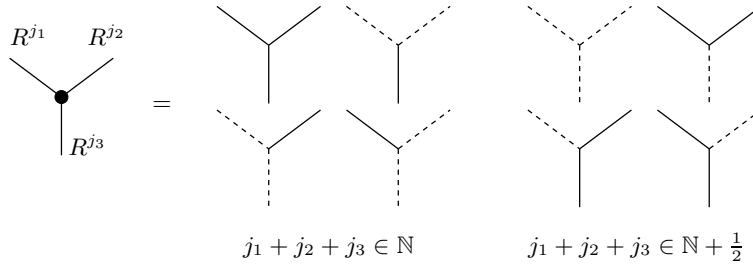


FIG. 1: The $\mathfrak{osp}_E(1|2)$ intertwiner. The isospin spaces V^{j_i} are denoted by solid lines and $V^{j_i - \frac{1}{2}}$ by dotted lines.

Ponzano-Regge state sum. The other four satisfy $j_1 + j_2 + j_3 \in \mathbb{N} + \frac{1}{2}$. These $\mathfrak{su}(2)$ intertwiners inherit their normalisation from that of the $\mathfrak{osp}_E(1|2)$ intertwiner. The interpretation proposed in [4] for these intertwiners is that the first four are bosonic and represent pure gravity, while the second four (which do not occur in the $\mathfrak{su}(2)$ Ponzano-Regge model) are fermionic intertwiners and denote the presence of a fermion. Importantly, the two $\mathfrak{osp}_E(1|2)$ intertwiners attached to the shared face of two adjacent tetrahedra are the same. Thus, they are both bosonic or both fermionic (although they need not be exactly the same bosonic or

⁶ A circuit diagram consists of lines called wires, and boxes called cables. The circuit diagram is an enrichment of the dual 2-skeleton Δ_2^* . Each edge e^* of Δ_2^* is replaced by a cable through which three wires pass. Each wire inherits the representation of one of the three incident faces f^* of Δ^* . The 4-valent vertices of Δ_2^* are replaced by a routing of the twelve incident wires, such that the faces f^* of Δ_2^* are replaced by a closed loop. The non-trivial rule which can now be encoded is that should two wires cross in the planar embedding of the circuit diagram, then one should include a factor of $(-1)^{\lambda_j \lambda_k}$ where the λ are the parities of the involved representations.

fermionic). Therefore, the fermions may be thought of as propagating along the edges of the dual 2-skeleton Δ_2^* .

Again, due to the isospin decomposition of the representations, we can distinguish between different types of tetrahedral amplitude, which are formed from a contraction of the intertwiners I_f assigned to the four triangles of a tetrahedron. There are many possible terms, but they fall into three classes: those consisting of four bosonic intertwiners (G, G, G, G) , two fermionic and two bosonic intertwiners (G, G, F, F) , and four fermionic intertwiners (F, F, F, F) . The cases with an odd number of fermionic intertwiners have zero amplitude. For example, we illustrate the class of diagrams (up to permutation) occurring in the (G, G, F, F) tetrahedron in Figure 2. The tetrahedra drawn in the diagram are not a space-time tetrahedra but each are the contraction of four intertwiners (the dual spin network). Note that dotted lines either join the fermion or form a closed loop. This is a generic feature of any boundary spin network and was proven in [4].

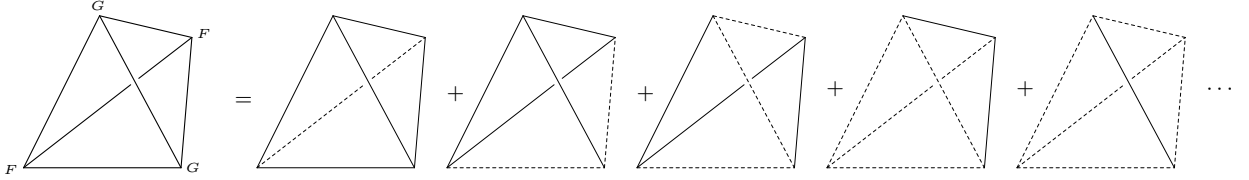


FIG. 2: The (G, G, F, F) class of diagrams. G stands for bosonic intertwiners, while F represents fermionic intertwiners. There are five diagrams occurring in this decomposition up to permutations.

Through this construction we have identified fermionic degrees of freedom attached to the intertwiners, and interpreted the $\mathfrak{osp}_E(1|2)$ Ponzano-Regge model as providing a path integral for gravity plus fermions. The fermionic edges of the dual 2-skeleton form a Feynman graph on the dynamical spin foam. The interpretation of the amplitudes given here is internally consistent but needs to be solidified by analysing, for example, the semiclassical regime in which one could relate the amplitudes to those coming from Regge calculus coupled to spin- $\frac{1}{2}$ fermions [15].

III. $N=2$ SUPERGRAVITY AND SUPERSYMMETRIC SPIN FOAMS

The analysis of Achúcarro and Townsend [5] extends the Chern-Simons formalism for three dimensional gravity to a whole class of supersymmetric cases. The symmetry algebra of Lorentzian Anti de Sitter gravity is $\mathfrak{sp}(2) \oplus \mathfrak{sp}(2)$. Replacing this by its supersymmetric counterpart $\mathfrak{osp}_L(p|2) \oplus \mathfrak{osp}_L(q|2)$, we arrive at theory known as (p, q) AdS supergravity. An analogous avenue can be followed for Euclidean de Sitter gravity where we replace $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ by $\mathfrak{osp}_E(p|2) \oplus \mathfrak{osp}_E(q|2)$.⁷

The theory in which we shall be interested is $(2, 2)$ Riemannian dS supergravity, that is $p = 2$ and $q = 2$. Similar to case of $N = 1$ Chern-Simons supergravity, we can write this theory as a BF -theory with cosmological constant term based on the gauge algebra $\mathfrak{osp}_E(2|2) \oplus \mathfrak{osp}_E(2|2)$. Taking the limit where the cosmological constant vanishes one arrives at the action

$$\mathcal{S}[\mathcal{E}, \mathcal{A}] = \int_{\mathcal{M}} \text{Str}(\mathcal{E} \wedge \mathcal{F}(\mathcal{A})). \quad (10)$$

The $N = 2$ supertriad and superconnection fields are defined as

$$\mathcal{E} = E^i J_i + \phi^A Q_A + \tilde{\phi}^{\tilde{A}} \tilde{Q}_{\tilde{A}} + eB, \quad \mathcal{A} = W^i J_i + \psi^A Q_A + \tilde{\psi}^{\tilde{A}} \tilde{Q}_{\tilde{A}} + wB, \quad (11)$$

where E and W are the $\mathfrak{su}(2)$ -valued triad and connection respectively. ϕ and ψ contain the degrees of freedom of the positively charged fermion, while $\tilde{\phi}$ and $\tilde{\psi}$ represent the negatively charged fermion. e is the

⁷ Note that the bosonic sector of $\mathfrak{osp}_L(p|2)$ is $\mathfrak{o}(p) \oplus \mathfrak{sp}(2)$ while the bosonic sector of $\mathfrak{osp}_E(p|2)$ is $\mathfrak{o}(p) \oplus \mathfrak{su}(2)$.

electric field for the $\mathfrak{u}(1)$ gauge theory while w is the $\mathfrak{u}(1)$ connection. Finally, $\mathcal{F}[\mathcal{A}] = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$. The Lie algebra elements $J_i, Q_\pm, \tilde{Q}_\pm, B$ satisfy the commutation relations:

$$\begin{aligned} [J_+, J_-] &= 2J_3, & [J_3, J_\pm] &= \pm J_\pm, & [J_\pm, B] &= 0, & [J_3, B] &= 0, \\ [B, Q_\pm] &= \frac{1}{2}Q_\pm, & [B, \tilde{Q}_\pm] &= -\frac{1}{2}\tilde{Q}_\pm, & [J_3, Q_\pm] &= \pm \frac{1}{2}Q_\pm, & [J_3, \tilde{Q}_\pm] &= \pm \frac{1}{2}\tilde{Q}_\pm, \\ [J_\pm, Q_\mp] &= Q_\pm, & [J_\pm, \tilde{Q}_\mp] &= \tilde{Q}_\pm, & [J_\pm, Q_\pm] &= 0, & [J_\pm, \tilde{Q}_\pm] &= 0, \\ \{Q_\pm, Q_\pm\} &= 0, & \{Q_\pm, Q_\mp\} &= 0, & \{\tilde{Q}_\pm, \tilde{Q}_\pm\} &= 0, & \{\tilde{Q}_\pm, \tilde{Q}_\mp\} &= 0, \\ & & \{Q_\pm, \tilde{Q}_\pm\} &= \pm J_\pm, & \{Q_\pm, \tilde{Q}_\mp\} &= -J_3 \pm B. \end{aligned} \quad (12)$$

The generators are of two types, bosonic and fermionic [6]. The bosonic sector is generated by: J_\pm, J_3, B ; it is an $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ subalgebra. $\mathfrak{su}(2)$ is often called isospin and $\mathfrak{u}(1)$ is the charge. Q_\pm and \tilde{Q}_\pm are fermionic generators.⁸

Written out in components of the multiplets \mathcal{E} and \mathcal{A} , the action can be rewritten as

$$\mathcal{S}_{N=2}[\mathcal{E}, \mathcal{A}] = \int_{\mathcal{M}} \left\{ \text{Str}(E \wedge (F[W] + \psi \wedge \tilde{\psi}) + e \wedge (f[w] + \psi \wedge \tilde{\psi}) + \tilde{\phi} \wedge D\psi + \phi \wedge D\tilde{\psi}) \right\} \quad (16)$$

where $D = \partial + W + w$ is the covariant derivative with respect to gravity and the $\mathfrak{u}(1)$ gauge theory. $F(W) = dW + W \wedge W$ is the gravitational curvature and $f(w) = dw$ is the $\mathfrak{u}(1)$ gauge curvature. We note here that the gauge theory is a $\mathfrak{u}(1)$ BF -theory rather than electrodynamics in three dimensions. Through this mechanism, the fermions acquire a charge but we do not see chargeless spin-1 particles propagating and interacting in our theory.

The state sum model will be a topological theory based on the representations $R^{j,b}$ of the superalgebra $\mathfrak{osp}_E(2|2)$. In the framework of spin foam models, it should implement a path integral for gravity plus charged fermions in three dimensions. We follow the same procedure as in Section II. The resulting amplitude is

$$\mathcal{Z}_{N=2} = \sum_{R^{j_e, b_e}} \prod_e \text{Sdim}(R^{j_e, b_e}) \prod_v \mathcal{T}(R^{j_e, b_e}) \quad (17)$$

There are several differences between the case of $\mathfrak{osp}_E(2|2)$ and $\mathfrak{osp}_E(1|2)$. We wish to assign a representation to each edge of Δ , but to add a complication, the representation theory is plagued by representations which are not completely reducible. More explicitly, the representations of $\mathfrak{osp}_E(2|2)$ may be written as the sum of representations of its bosonic subalgebra $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$. The representations $V^{j,b}$ of $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ are labeled by an isospin $j \in \frac{1}{2}\mathbb{N}$ and a charge $b \in \mathbb{C}$.

The representations fall into four categories

- Typical irreducible - $j \neq \pm b$: The typical representations consist of four multiplets of the bosonic subalgebra $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$

$$R^{j,b} = V^{j,b} \oplus V^{j-\frac{1}{2}, b-\frac{1}{2}} \oplus V^{j-\frac{1}{2}, b+\frac{1}{2}} \oplus V^{j-1, b}.$$

⁸ The map

$$J_i \rightarrow J_i, \quad B \rightarrow -B, \quad Q_\pm \rightarrow \tilde{Q}_\pm, \quad \tilde{Q}_\pm \rightarrow Q_\pm \quad (13)$$

is an automorphism of the algebra. The quadratic Casimir for the algebra is given by

$$C_2 = J^2 - B^2 + \frac{1}{2}(Q_+ \tilde{Q}_- - Q_- \tilde{Q}_+ - \tilde{Q}_+ Q_- - \tilde{Q}_- Q_+). \quad (14)$$

Thus, the inner product on the algebra is given by

$$\text{tr}(J_i J_j) = \delta_{ij}, \quad \text{tr}(BB) = -1, \quad \text{tr}(Q_+ \tilde{Q}_-) = -2, \quad \text{tr}(\tilde{Q}_+ Q_-) = -2 \quad (15)$$

There also exists a cubic Casimir C_3 .

where $V^{j,b}$ is the representation of $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ with quantum numbers j and b referring to $\mathfrak{su}(2)$ and $\mathfrak{u}(1)$, respectively. The dimension of such a representation is $8j$.

- Atypical irreducible - $j = b$: The atypical representations consist of two multiplets of $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$

$$R^{j,j} = V^{j,j} \oplus V^{j-\frac{1}{2},j+\frac{1}{2}}.$$

The dimension of the representation is $4j + 1$.

- Atypical irreducible - $j = -b$:

$$R^{j,j} = V^{j,-j} \oplus V^{j-\frac{1}{2},-j-\frac{1}{2}}.$$

The dimension of such a representation is again $4j + 1$.

- Atypical not-completely reducible - $j = \pm b$: There are also reducible representations of dimension $8j$ for $j = \pm b$, consisting of four multiplets of $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$, which contain the atypical irreducible representation as an invariant subspace.

$$R^{j,b} = V^{j,b} \oplus V^{j-\frac{1}{2},b-\frac{1}{2}} \oplus V^{j-\frac{1}{2},b+\frac{1}{2}} \oplus V^{j-1,b}.$$

The complementary subset is not itself an invariant subspace, so the representation is not fully reducible.

The Casimirs C_2 and C_3 do not classify the atypical representations. Evaluating these operators on the representations, we arrive at $C_2 = j^2 - b^2$ and $C_3 = b(j^2 - b^2)$, meaning that both are zero on all such representations. The action of the algebra is given in Appendix A.

Our first thought might be to simply exclude the troublesome atypical representations from the beginning. But we must be more careful than that since these representations can occur in the decomposition of the tensor product of two typical representations

$$R^{j_1,b_1} \otimes R^{j_2,b_2} = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} R^{j,b_1+b_2} \oplus \bigoplus_{j=|j_1-j_2+\frac{1}{2}|}^{j_1+j_2-\frac{1}{2}} R^{j,b_1+b_2+\frac{1}{2}} \oplus \bigoplus_{j=|j_1-j_2+\frac{1}{2}|}^{j_1+j_2-\frac{1}{2}} R^{j,b_1+b_2-\frac{1}{2}} \oplus \bigoplus_{j=|j_1-j_2-1|}^{j_1+j_2+1} R^{j,b_1+b_2} \quad (18)$$

where the sums over j are in *integer* steps. If $b_1 = j_1 - 1$ and $b_2 = j_2 - 1$, we see that an atypical representations occurs in the first sum on the right hand side.

Therefore, we must place an extra condition on the class of representations allowed to label the edges of our simplicial complex. It turns out that the subset of representations satisfying $\pm b > j$ forms a subcategory of representations closed under tensor product. We can also motivate this condition by noticing that it is exactly these representations upon which we can define a star operator (with respect to a positive definite scalar product)[6]. In fact, a grade star operator does not exist on the representations of $\mathfrak{osp}_E(2|2)$ (apart from two special cases). Therefore, the parity considerations, which we had to deal with in the $N = 1$ case, are no longer present.⁹ In fact, the conditions for the existence of a star operator are stronger still. It requires $b \in \mathbb{R}$, and that the scalar product reduce to the usual one when restricted to $\mathfrak{su}(2)$. The effect of the adjoint operation on the algebra is

$$\begin{aligned} &\text{For all } \pm b > j, \quad J_i^\dagger = J_i, \quad B^\dagger = B, \\ &\text{for } b > j, \quad Q_+^\dagger = \tilde{Q}_-, \quad Q_-^\dagger = -\tilde{Q}_+, \quad \tilde{Q}_+^\dagger = Q_-, \quad \tilde{Q}_-^\dagger = -Q_+, \\ &\text{for } b < -j, \quad Q_+^\dagger = -\tilde{Q}_-, \quad Q_-^\dagger = \tilde{Q}_+, \quad \tilde{Q}_+^\dagger = -Q_-, \quad \tilde{Q}_-^\dagger = Q_+, \end{aligned} \quad (19)$$

Interestingly, the Bogomol'nyi bound for this algebra is $\{Q_A, \tilde{Q}_B\} \{Q^A, \tilde{Q}^B\} = 2(BB - J^i J_i) \geq 0$. This imposes the condition $\pm b \geq \sqrt{j(j+1)}$. This is in fact a slightly stronger condition than what we have

⁹ This means that the superdimension of an $\mathfrak{osp}_E(2|2)$ representation is the same as the dimension.

already imposed, and means that the BPS states (which saturate this bound) lie within the restricted subcategory of representations.

Working with this subcategory of representations closed under tensor product allows to define a 3d topological state sum based on $\mathfrak{osp}_E(2|2)$ following the framework introduced in [4, 7]. This is our proposal for the spinfoam quantization of $N = 2$ 3d supergravity.

Moreover, this analysis can be applied to $\mathfrak{osp}_E(2|2)$ spin network states for $N = 2$ loop quantum supergravity in the usual $3 + 1$ spacetime dimensions [3]. In this context, we can define a notion of BPS states saturating the proposed $|b| \geq j$ bound. It would be interesting to investigate further what consequences this has on the quantum states describing a supersymmetric black hole in loop gravity.

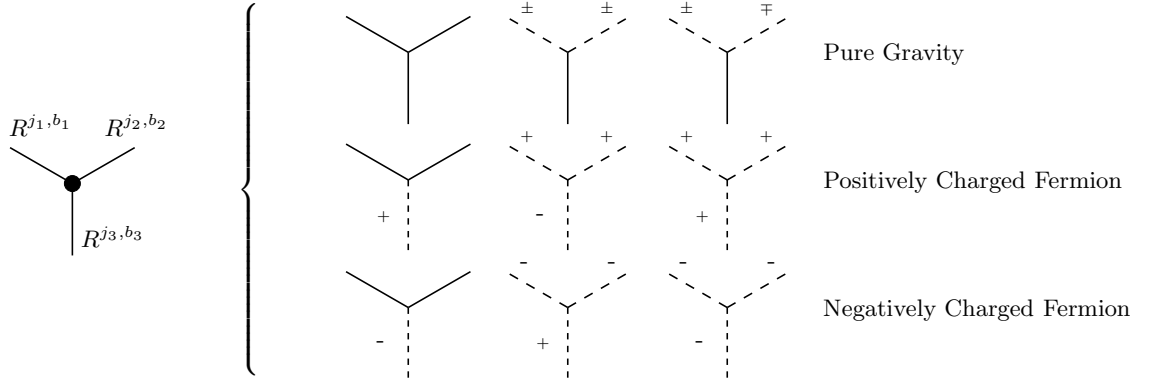


FIG. 3: The $\mathfrak{osp}_E(2|2)$ intertwiner. The $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ representations $V^{j,b}$ and $V^{j-1,b}$ are denoted by solid lines, the $V^{j-\frac{1}{2}, b-\frac{1}{2}}$ by $+$ -dashed lines, and the $V^{j-\frac{1}{2}, b+\frac{1}{2}}$ by the $-$ -dashed lines. We do not draw all the diagrams; permutations of the above are possible.

To construct the amplitude, we need to label each edge with an orientation. Then for the each face of every tetrahedron, we assign an intertwiner $I_f : R^{j_1, b_1} \otimes R^{j_2, b_2} \otimes R^{j_3, b_3} \rightarrow \mathbb{C}$ or $I_f : R^{j_1, b_1} \otimes R^{j_2, b_2} \otimes (R^{j_3, b_3})^* \rightarrow \mathbb{C}$ depending on the orientation of the edges. The intertwiner is unique up to normalization and we normalize as before. To construct the tetrahedral amplitude we contract four such intertwiners.

We illustrate the intertwiner $I_f : R^{j_1, b_1} \otimes R^{j_2, b_2} \otimes R^{j_3, b_3} \rightarrow \mathbb{C}$ in Figure 3. Once again, the decomposition of the representations allows one to distinguish subclasses within the $\mathfrak{osp}_E(2|2)$ intertwiner. Our rationale when classifying these diagrams is that those marked as pure gravity satisfy $j_1 + j_2 + j_3 \in \mathbb{N}$. The other classes satisfy $j_1 + j_2 + j_3 \in \mathbb{N} + \frac{1}{2}$ and $b_1 + b_2 + b_3 \in \mathbb{Z} + \frac{1}{2}$. In particular, the positively charged fermionic intertwiner has $b_1 + b_2 + b_3 > 0$, while negatively charged one has $b_1 + b_2 + b_3 < 0$. The fermions propagate along edges of the spin foam Δ_2^* .

The tetrahedral amplitude is a $\{6j, 6b\}_{\mathfrak{osp}(2|2)}$ -symbol meaning that each term in the sum is the product of a $\{6j\}_{\mathfrak{su}(2)}$ and a $\{6b\}_{\mathfrak{u}(1)}$. We separate its constituent diagrams into nine classes: (G, G, G, G) , (G, G, F^+, F^+) , (G, G, F^+, F^-) , (G, G, F^-, F^-) , (F^+, F^+, F^+, F^+) , (F^+, F^+, F^+, F^-) , (F^+, F^+, F^-, F^-) , (F^+, F^-, F^-, F^-) , (F^-, F^-, F^-, F^-) , where G stands for a bosonic intertwiner, F^\pm stands for a \pm -charged fermionic intertwiner. We illustrate one class of diagrams in Figure 4. It is interesting to note that there are such terms as (G, G, F^+, F^-) . As we sum over both orientations, this language represents two processes. The first is as the annihilation of an oppositely charged fermion pair. The second is as the interaction of a positively charged fermion with the dynamical “background” $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ gauge field to produce a negatively charged fermion.

Finally, the amplitude (17) represents the propagation of charged particles in a dynamical geometry. The fermion paths form a Feynman graph embedded into the spin foam. Topologically, the Feynman graphs are the same as in the $N = 1$ case, but here they contain more information. The particles are charged under an extra $\mathfrak{u}(1)$ algebra. Essentially the particles only propagate, there only interaction is with the gauge field in order . This mimics the classical action (which is cubic and so could not contain the interaction of spin- $\frac{1}{2}$ fermions) where one notices the *gauge – fermion – antifermion* interaction terms.

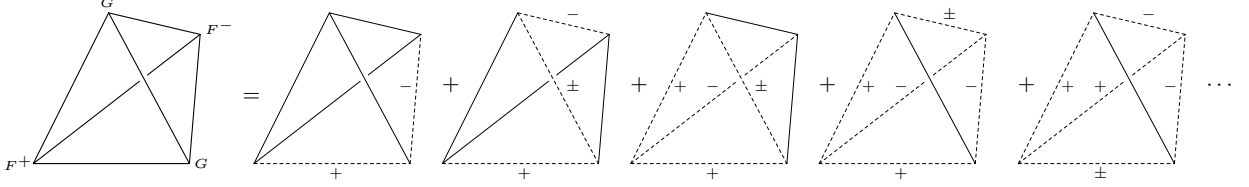


FIG. 4: The (G, G, F^+, F^-) class of diagrams. G stands for the bosonic intertwiners, while F^\pm represents the \pm -fermionic intertwiners. There are five diagrams occurring in this decomposition up to permutations.

A. The Lorentzian Case

Up to this point we have only described the case of $3d$ Riemannian gravity, but a more physically interesting case is $2+1-d$ gravity where the underlying symmetry algebra is $\mathfrak{su}(1, 1)$. In the supersymmetric context, we examine the case of a spin foam model based on $\mathfrak{osp}_L(2|2)$ where the bosonic subalgebra is $\mathfrak{su}(1, 1) \oplus \mathfrak{u}(1)$.¹⁰ We arrive at the generators for this algebra through rotating by i the generators J_\pm of its Riemannian counterpart $\mathfrak{osp}_E(2|2)$.¹¹ The only changes in the commutation relations (12) are

$$[J_\pm, Q_\mp] = -iQ_\pm, [J_\pm, \tilde{Q}_\mp] = -i\tilde{Q}_\pm, \{Q_\pm, \tilde{Q}_\pm\} = \pm iJ_\pm. \quad (20)$$

The representations of $\mathfrak{osp}_L(2|2)$ can once again be decomposed into a direct sum of representations of its bosonic subalgebra. We restrict to the unitary principal representations of $\mathfrak{su}(1, 1)$ which are of two kinds: a continuous series labeled by $s \in \mathbb{R}$ and two discrete series labeled by $j \in \frac{1}{2}\mathbb{N}$, one positive and one negative. The basis for the continuous series is the set of vectors $|s, j_3\rangle$ where either $j_3 \in \mathbb{Z}$ or $j_3 \in \mathbb{Z} + \frac{1}{2}$. The basis for the positive discrete series is the set of vectors $|j, j_3\rangle$ with $j_3 \geq j$ and $j_3 \in \frac{1}{2}\mathbb{N}$, while for the negative series it is the set of vectors $|j, j_3\rangle$ with $j_3 \leq -j$ and $-j_3 \in \frac{1}{2}\mathbb{N}$. The Casimir evaluated on these representations (in its capacity kinematical length-squared operator) suggests that the continuous representations should label space-like edges while the positive and negative discrete series should label the future time-like and past time-like edges respectively [16].¹² In $2+1-d$ quantum gravity, we have a choice when proposing a topological state sum: we can choose to sum over all the principal representations, or we can restrict the sum to just the positive discrete series. Interestingly, the set of positive discrete representations is a subcategory closed under tensor product. A simplicial complex labeled in either way will be a topological invariant. We will restrict to this class of representations for the $\mathfrak{su}(1, 1)$ subalgebra of $\mathfrak{osp}_L(2|2)$ also. With this proviso, the representation theory of $\mathfrak{osp}_L(2|2)$ has a very similar appearance to that of $\mathfrak{osp}_E(2|2)$. The representations are $R^{j,b}$ where j is the isospin and $b \in \mathbb{C}$ is the charge. They once again fall into four categories, one typical and three atypical. Under tensor product however, the representations satisfy

$$R^{j_1, b_1} \otimes R^{j_2, b_2} = \bigoplus_{j > j_1 + j_2} R^{j, b_1 + b_2} \oplus \bigoplus_{j > j_1 + j_2 - \frac{1}{2}} R^{j, b_1 + b_2 + \frac{1}{2}} \oplus \bigoplus_{j > j_1 + j_2 - \frac{1}{2}} R^{j, b_1 + b_2 - \frac{1}{2}} \oplus \bigoplus_{j > j_1 + j_2 + 1} R^{j, b_1 + b_2} \quad (21)$$

This time around, the restriction to a consistent subcategory of representations is $|b| < j$, which is simply the reverse inequality compared to the Riemannian theory.

Now that we have successfully defined a consistent representation theory, we can proceed in an analogous fashion to the Riemannian scenario and define a spin foam model based on the restricted subcategory of representations. The state sum will be triangulation independent and by decomposing the representations into their isospins, we can interpret certain terms in the sum as representing charged fermions along edges of the spin foam.

¹⁰ The $N = 1$ case was dealt with in [4].

¹¹ The Hermiticity relations for $\mathfrak{su}(1, 1)$ are $J_\pm^\dagger = -J_\mp$ and $J_3^\dagger = J_3$.

¹² The Casimir for $\mathfrak{su}(1, 1)$ takes the values: $s^2 + \frac{1}{4}$ for the continuous series and $-j(j-1)$ for the discrete series. They are of opposite sign.

B. An alternative state sum construction

The fact that a $\mathfrak{osp}_E(1|2)$ irreducible representation is made of two $\mathfrak{su}(2)$ spin representations suggests another strategy to derive spinfoam models with higher supersymmetry. One could try to pile $\mathfrak{su}(2)$ representations together in order to form supersymmetric multiplets. For example, we can consider the reducible representations made as the direct sum of three copies of $\mathfrak{su}(2)$,

$$R^j = V^j \oplus V^{j-\frac{1}{2}} \oplus V^{j-1} \quad (22)$$

Indeed, this satisfies the relation

$$R^j \otimes R^k = \bigoplus_{l=|j-k|}^{j+k} R^l \oplus \bigoplus_{l=|j-k+\frac{1}{2}|}^{j+k-\frac{1}{2}} R^l \oplus \bigoplus_{l=|j-k+1|}^{j+k-1} R^l \quad (23)$$

under tensor product. This category of representations is closed under tensor product, so we can build a topological state sum based on them. This actually generalizes to stacks of $\mathfrak{su}(2)$ representations of arbitrary size.

On the other hand, the task now is to identify an algebra to which the $R^{(j)}$'s provide faithful (irreducible) representations. Since they are made of three $\mathfrak{su}(2)$ representations, it could be interpreted as the representation of some algebra in between $N = 1$ and $N = 2$ supergravity theories. Actually, it seems to correspond to the $\mathfrak{osp}_E(2|2)$ algebra where we would have gauge-fixed the $\mathfrak{u}(1)$ generator B to $b = 0$. Then the quadratic Casimir of the resulting algebra is $C_2 = \eta^{ij} J_i J_j + Q_A \epsilon^{AB} \tilde{Q}_B$ which yields $C_2 = j^2$ on a representation. This characterizes the representation uniquely.

The next difficulty is to find the classical theory from which one could derive this model. Our guess is simply

$$\mathcal{S}_{alt}[\mathcal{E}, \mathcal{A}] = \int_{\mathcal{M}} \text{Str}(\mathcal{E} \wedge \mathcal{F}[\mathcal{A}]), \quad (24)$$

where we have removed the $\mathfrak{u}(1)$ component of the superfields,

$$\mathcal{E} = E^i J_i + \phi^A Q_A + \tilde{\phi}^A \tilde{Q}_A, \quad \mathcal{A} = W^i J_i + \psi^A Q_A + \tilde{\psi}^A \tilde{Q}_A, \quad (25)$$

Written out explicitly in components of the multiplets \mathcal{E} and \mathcal{A} , this action can be rewritten as

$$\mathcal{S}_{alt}[\mathcal{E}, \mathcal{A}] = \int_{\mathcal{M}} \left\{ \text{Str}(E \wedge (F[W] + \psi \wedge \tilde{\psi}) + \tilde{\phi} \wedge D\psi + \phi \wedge D\tilde{\psi}) \right\} \quad (26)$$

where $D = \partial + W \wedge$ is the covariant derivative with respect to gravity and $F(W) = dW + W \wedge W$ is the gravitational curvature. This spin foam model would then give configurations for gravity plus two indistinguishable fermion types.

IV. CONCLUSION

In this paper, we reviewed the relevant aspects of $N = 1$ quantum supergravity in the three dimensions and extended the theory to the $N = 2$ case. This is a richer theory as it contains a $\mathfrak{u}(1)$ gauge theory. We constructed a spin foam based on the gauge algebra $\mathfrak{osp}(2|2)$. As with $N = 1$ supergravity, the classical action contains fermionic degrees of freedom. We must identify these properties in the quantum theory. The super Ponzano-Regge state sum contains configurations that do not arise in the $\mathfrak{su}(2)$ quantum gravity state sum. More precisely, the difference is that in the supergravity theories, the triple of isospins labeling the three edges of a triangle need not necessarily satisfy $j_1 + j_2 + j_3 \in \mathbb{N}$. The intertwiner of such a triple is viewed as a spin- $\frac{1}{2}$ fermion propagating along an edge of the spin foam. When the gauge algebra is $\mathfrak{osp}(1|2)$ there is one type of fermion, while for the $\mathfrak{osp}(2|2)$, we have positively and negatively charged fermions.

This interpretation is not steadfast, however, and we need to solidify our reasoning with an analysis of the pertinent semi-classical limit. This will be the subject of later work [15]. We shall develop a connection between the model developed here and fermionic fields coupled to gravity in the arena of Regge calculus. The strategy will be to analyze the asymptotic behavior of the $N = 1$ and $N = 2$ supersymmetric $\{6j\}$ symbols. Indeed, we already know how the standard $\{6j\}$ symbol for $\mathfrak{su}(2)$ is related to the Regge action for 3d gravity. Factorizing out this term in the supersymmetric $\{6j\}$ symbols, we hope to identify the discrete path integral amplitude describing the dynamics of the supersymmetric fields coupled to gravity. Within the spinfoam graviton propagator framework [17], this will allow to compute the scattering amplitudes for fermions and $u(1)$ gauge fields (for $N = 2$) coupled to 3d quantum gravity.

One should also study whether it is possible to extend the state sum in the Lorentzian case to include the continuous representations, and to check the conditions required to allow the existence of a star operator. The continuous representations label space-like edges and so are necessary to define a spatial hypersurface.

The extension to higher BF-theory could be seen as the first link in the chain connecting this work to supergravity, which occurs as a constrained BF theory [2], and with a possible non-perturbative background independent definition of M-theory following the logic of [18].

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APPENDIX A: ACTION ON A TYPICAL REPRESENTATION

The action of the operators on the representations is (for $R^{j,b}$, $j \neq \pm b$):

$$J_{\pm}|j, j_3, b\rangle = \sqrt{(j \mp j_3)(j \pm j_3 + 1)}|j, j_3 \pm 1, b\rangle, \quad J_3|j, j_3, b\rangle = j_3|j, j_3, b\rangle, \quad B|j, j_3, b\rangle = b|j, j_3, b\rangle,$$

$$Q_{\pm}|j, j_3, b\rangle = \pm \sqrt{\frac{(b+j)(j \mp j_3)}{2j}}|j - \frac{1}{2}, j_3 \pm \frac{1}{2}, b + \frac{1}{2}\rangle$$

$$\tilde{Q}_{\pm}|j, j_3, b\rangle = \pm \sqrt{\frac{(b-j)(j \mp j_3)}{2j}}|j - \frac{1}{2}, j_3 \pm \frac{1}{2}, b - \frac{1}{2}\rangle$$

$$Q_{\pm}|j - \frac{1}{2}, j_3, b - \frac{1}{2}\rangle = \sqrt{\frac{(b-j)(j \pm j_3 + \frac{1}{2})}{2j}}|j, j_3 \pm \frac{1}{2}, b\rangle \mp \sqrt{\frac{(b+j)(j \mp j_3 - \frac{1}{2})}{2j}}|j - 1, j_3 \pm \frac{1}{2}, b\rangle$$

$$\tilde{Q}_{\pm}|j - \frac{1}{2}, j_3, b - \frac{1}{2}\rangle = 0$$

$$Q_{\pm}|j - \frac{1}{2}, j_3, b + \frac{1}{2}\rangle = 0$$

$$\tilde{Q}_{\pm}|j - \frac{1}{2}, j_3, b + \frac{1}{2}\rangle = \sqrt{\frac{(b+j)(j \pm j_3 + \frac{1}{2})}{2j}}|j, j_3 \pm \frac{1}{2}, b\rangle \pm \sqrt{\frac{(b-j)(j \mp j_3 - \frac{1}{2})}{2j}}|j - 1, j_3 \pm \frac{1}{2}, b\rangle$$

$$Q_{\pm}|j - 1, j_3, b\rangle = \sqrt{\frac{(b-j)(j \pm j_3)}{2j}}|j - \frac{1}{2}, j_3 \pm \frac{1}{2}, b + \frac{1}{2}\rangle$$

$$\tilde{Q}_{\pm}|j - 1, j_3, b\rangle = -\sqrt{\frac{(b+j)(j \pm j_3)}{2j}}|j - \frac{1}{2}, j_3 \pm \frac{1}{2}, b - \frac{1}{2}\rangle$$

The value assumed by the Casimirs acting on each state of the representation $R^{j,b}$ are:

$$C_2 : \quad j^2 - b^2 \quad C_3 : \quad b(j^2 - b^2). \quad (\text{A.1})$$

It is clear that they both vanish for $j = \pm b$, so they do not pick out these representations. The states are normalised.

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